Chapter 5

Fourier Series on Compact Groups

In the previous chapters we saw we could identify functions on a homogeneous space (X, G) with a set X and transformation group G with functions on the group G. The most important examples are for us the unit circle together with $SO(2)$ and the unit sphere together with $SO(3)$. In the last chapter (theorem 3.13 and the examples in 3.3) we saw that the irreducible representations of $SO(2)$ are given by the functions $e^{im\phi}$. From the elementary theory of Fourier series it is known that every function on the unit circle can be developed into a Fourier series. Reformulating this in group theoretical language this gives:

"Every function on the homogeneous space X (the unit circle) can be developed into a series of matrix entries of the irreducible representations of $G(= SO(2))$."

In Hilbert space terminology this becomes:

"The matrix entries of the irreducible representations of G form a basis of the L^2 space $L^2(X)$."

In this chapter we will mainly prove that this statement is true for all compact groups. In the second section of this chapter we will then use this result to show some fundamental properties of the characters of a group.

5.1 The Peter-Weyl Theorem

We define the space of all square-integrable functions on a compact group as follows:

Definition 5.1 Let G be a compact group with Haar integral $\int_G f(g)\, dg$. Then we define $L^2(G)$ as the set of complex-valued functions f that satisfy the following conditions:

1. f is measurable under the Haar measure
2. $\int_g |f(g)|^2 \, dg < \infty$

On $L^2(G)$ we define the scalar product:

$$\langle f_1, f_2 \rangle = \int_G f_1(g)\overline{f_2(g)}\, dg$$

With these definitions $L^2(G)$ becomes a Hilbert space.

In this space we define the regular representations of G as:

Definition 5.2 1. The representation $T : G \to L^2(G); h \mapsto T(h)$ with

$$T(h)f(g) = f(gh)$$

is called the *right regular representation* of G.

2. The representation $S : G \to L^2(G); h \mapsto S(h)$ with

$$S(h)f(g) = f(h^{-1}g)$$

is called the *left regular representation* of G.

To see that the definitions are meaningful we have to show that $T(h)f$ is indeed an element of $L^2(G)$ and that T is linear and continuous. We omit the proof since it is purely technical.

Theorem 5.1 The left- and the right-regular representations are unitary.

This can be seen by a simple computation:

$$\langle T(h)f_1, T(h)f_2\rangle = \int_G f_1(gh)\overline{f_2(gh)}\, dg = \int_G f_1(g)\overline{f_2(g)}\, dg = \langle f_1, f_2\rangle$$

Theorem 5.2 The left- and the right regular representations of G are unitarily equivalent, i.e. there is a one-to-one mapping $A : L^2(G) \to L^2(G)$ such that A is unitary and $AS(g) = T(g)A$ for all $g \in G$.

Define $Af(g) = f(g^{-1})$ and find for $TA(g)$:

$$\begin{aligned} T(g)Af(h) &= T(g)f(h^{-1}) = f(h^{-1}g) = f((g^{-1}h)^{-1}) && (5.1)\\ &= A(f(g^{-1}h)) = A(S(g)f(h)) && (5.2) \end{aligned}$$

We now investigate the matrix entries of finite-dimensional, irreducible, unitary representations of a compact group G. In the rest of this section we will assume that the following conditions are satisfied:

We assume that the group G is compact, by T we denote a finite-dimensional, irreducible, unitary representation of G in a Hilbert space H. We fix an orthonormal basis in H and denote the elements of this basis by $e_1, ..., e_n, ...$. In this basis $T(g)$ is described by a unitary matrix and we will also denote this matrix by $T(g)$. The matrix elements of T are denoted by $t_{ij}(g)$ and as functions on G they are continuous. If we want to investigate different representations then we use a superscript to distinguish them. T^α is thus the representation belonging to the index α and $t^\alpha_{ij}(g)$ are the matrix entries of T^α. The dimension of the representation T^α will be denoted by $n_\alpha = \dim T^\alpha$. If A is some index set such that for all elements $\alpha \in A$ we have a finite, irreducible, unitary representation T^α then we say that the set $\{T^\alpha | \alpha \in A\}$ is a *complete set of finite-dimensional representations* if all finite-dimensional, irreducible, unitary representations of G are equivalent to one of the elements in $\{T^\alpha | \alpha \in A\}$. For the matrix entries of finite-dimensional, irreducible, unitary representations we have the following theorem:

Theorem 5.3 1. If T^α and $T^{\alpha'}$ are two finite-dimensional, irreducible, unitary representations then we have the following relations for their matrix entries:

$$\int_G t^\alpha_{ij}(g)t^{\alpha'}_{kl}(g)\,dg = \begin{cases} 0 & \text{if } i \neq k \text{ or } j \neq l \text{ or } \alpha \neq \alpha' \\ 1/n_\alpha & \text{if } i = k \text{ and } j = l \text{ and } \alpha = \alpha' \end{cases} \tag{5.3}$$

The relations in equation 5.3 are called the *orthogonality relations for a compact group.*

2. The functions $e^\alpha_{ij}(g) = \sqrt{n_\alpha}t^\alpha_{ij}(g)$ with $\alpha \in A, 0 \leq i,j \leq n_\alpha$ form an orthonormal system in $L^2(G)$.

Proof: Assume H and H' are the representation spaces of T^α and $T^{\alpha'}$. Denote further the space of all linear mappings from H to H' by L. For an operator $K \in L$ we define the two new operators:

$$K_1(g) = T^\alpha(g)KT^{\alpha'}(g^{-1})$$

and

$$K_2 = \int_G K_1(g)\,dg = \int_G T^\alpha(g)KT^{\alpha'}(g^{-1})\,dg$$

The operators K_1 and K_2 are also elements of L and we have furthermore

$$T^\alpha(h)K_2 = K_2T^{\alpha'}(g)$$

since

$$\begin{aligned} T^\alpha(h)K_2 &= T^\alpha(h)\int_G K_1(g)\,dg \\ &= T^\alpha(h)\int_G T^\alpha(g)KT^{\alpha'}(g^{-1})\,dg \\ &= \int_G T^\alpha(h)T^\alpha(g)KT^{\alpha'}(g^{-1})T^{\alpha'}(h^{-1})\,dgT^{\alpha'}(h) \\ &= \int_G T^\alpha(hg)KT^{\alpha'}((hg)^{-1})\,dgT^{\alpha'}(h) \\ &= \int_G T^\alpha(g)KT^{\alpha'}((g)^{-1})\,dgT^{\alpha'}(h) \\ &= \int_G K_1(g)\,dgT^{\alpha'}(h) = K_2T^{\alpha'}(h) \end{aligned} \tag{5.4}$$

If T^α and $T^{\alpha'}$ are two irreducible, inequivalent representations then we get from Schur's lemma 3.4 that $K_2 = 0$, i.e.

$$\int_G T^\alpha(g)KT^{\alpha'}(g^{-1})\,dg = 0 \tag{5.5}$$

for all linear maps K. Rewrite the last equation 5.5 with the matrix entries and get:

$$\int_G \sum_{\mu=1}^{n_\alpha}\sum_{\nu=1}^{n_{\alpha'}} t^\alpha_{j\mu}(g)k_{\mu\nu}t^{\alpha'}_{\nu i}(g^{-1})\,dg = 0 \tag{5.6}$$

for all $j = 0,\ldots,n_\alpha$ and all $i = 0,\ldots,n_{\alpha'}$ Using the identity: $t^{\alpha'}_{\nu i}(g^{-1}) = \overline{t^{\alpha'}_{i\nu}(g)}$ equation 5.6 becomes:

$$\int_G \sum_{\mu=1}^{n_\alpha}\sum_{\nu=1}^{n_{\alpha'}} t^\alpha_{j\mu}(g)k_{\mu\nu}\overline{t^{\alpha'}_{i\nu}(g)}\,dg = 0 \tag{5.7}$$

For the function $k_{\mu\nu}$ defined as

$$k_{\mu\nu} = \begin{cases} 1 & \text{for } \mu = m \text{ and } \nu = n \\ 0 & \text{for all other indices} \end{cases} \tag{5.8}$$

we get $\left\langle t^{\alpha}_{jm}(g), t^{\alpha'}_{in}(g)\right\rangle = 0$. This shows that the matrix entries are orthogonal if the representations are inequivalent. Now assume that $T^{n_\alpha} = T^{n'_\alpha}$ and write T instead of T^{n_α} and n instead of n_α. In this case we have $T(h)K_2 = K_2T(g)$ and from Schur's lemma we get $K_2 = \lambda id$. From the property of the trace we find:

$$\begin{aligned} \text{tr } K_2 &= \text{tr } \textstyle\int_G T(g)KT(g^{-1})\, dg \\ &= \text{tr } \textstyle\int_G T(g)K(T(g))^{-1}\, dg \\ &= \int_G \text{tr } T(g)K(T(g))^{-1}\, dg \\ &= \int_G \text{tr } K\, dg = \text{tr } K \end{aligned} \tag{5.9}$$

But since $K_2 = \lambda id$ we find tr $K_2 = \lambda \cdot n$ and therefore $\lambda = \text{tr } K_2/n$ and $K_2 = \lambda id = \frac{\text{tr } K_2}{n} id$. Selecting the operators K as in the equation 5.8 we find the expressions in 5.3 for the case where $\alpha = \alpha'$.

In the main theorem of this section we show that the matrix entries define not only an orthonormal system in $L^2(G)$ but that this system is also complete:

Theorem 5.4 [Peter-Weyl] Let the functions $e^{\alpha}_{ij}(g)$ be defined as in theorem 5.3. Then the system $\left\{e^{\alpha}_{ij}(g)\right\}_{\alpha}$ is complete in $L^2(G)$.

Proof: From the theory of Hilbert spaces it is known that we have to show that for every element $f(g) \in L^2(G)$ and every constant $\epsilon > 0$ there is a finite number of matrix entries $t^{\alpha}_{ij}(g)$ such that

$$\left\| f(g) - \sum_{i,j,\alpha} \gamma^{\alpha}_{ij} t^{\alpha}_{ij}(g) \right\| < \epsilon.$$

Take a real-valued continuous function χ, not identically equal to zero, such that $\chi(g^{-1}) = \chi(g)$ (for example $\chi(g) + \chi(g^{-1})$ for any non-negative function χ) and define $K(g_1, g_2) = \chi(g_1 g_2^{-1})$. This is a symmetric continuous function on $G \times G$ and we define the operator k as

$$\phi(g) \mapsto k(\phi)(g) = \psi(g) = \int_G K(g, g_1)\phi(g_1)\, dg_1 \tag{5.10}$$

The kernel K is quadratically integrable:

$$\int_{G\times G} |K(g,h)|^2\, dgdh < \infty \tag{5.11}$$

and from the Hilbert-Schmidt theory of integral equations we find that every function ψ of the form $\psi = k(\phi)$ is the sum of an absolutely and uniformly convergent series of eigenfunctions of the eigenvalue problem

$$k(\phi)(g) = \int_G K(g, g_1)\phi(g_1)\, dg_1 = \lambda\phi(g). \tag{5.12}$$

From the Hilbert-Schmidt theory it is also known that k has it least one eigenvalue λ and that the space $M_\lambda \subset L^2(G)$ consisting of the zero function and all eigenfunctions belonging to the eigenvalue λ forms a subspace of $L^2(G)$. This subspace is invariant under right translations. This can be easily seen by inserting $\phi(gg_0)$ into the equation 5.12 and computing the integral using the definition of K and the right invariance of the Haar integral.

M_λ is thus the representation space of a subrepresentation T_λ of the right regular representation. T_λ is continuous, finite-dimensional and unitary and can therefore be broken up into a finite number of irreducible unitary representations $T_\lambda^{(1)}, \ldots, T_\lambda^{(p)}$ and M_λ is the direct sum of the representation spaces $M_\lambda^{(1)}, \ldots, M_\lambda^{(p)}$).

We show now that all functions in $M_\lambda^{(k)}$ are linear combinations of the matrix entries of the representation $T_\lambda^{(k)}$:

Select a set of basis functions $e_i(g), i = 1, \ldots n$ in $M_\lambda^{(k)}$ and denote the matrix entries of the representation $T_\lambda^{(k)}$ by $c_{ij}(g)$. Since $T_\lambda^{(k)}$ is a subrepresentation of the right regular representation we have $T_\lambda^{(k)}(g_0)f(g) = f(g_0 g)$ for all $f \in M_\lambda^{(k)}$ and especially for the basis elements $e_i(g)$. Taking the identity $e \in G$ as g_0 we find that for every element $g \in G$ we have

$$e_i(g) = e_i(eg) = \sum_{k=0}^{n} c_{ki}(g) e_k(e). \tag{5.13}$$

The $e_k(e)$ are constants and the basis elements $e_i(g)$ are therefore linear combinations of the matrix entries c_{ij}. All elements in $M_\lambda^{(k)}$ are thus linear combinations of the matrix entries and we have thus shown that every element $\psi \in L^2(G)$ of the form 5.10 is a linear combination of the $c_{ij}(g)$.

We show now that every continuous function on G can be uniformly approximated by functions of the form 5.10.

We only sketch the proof: For a given continuous function f and a constant $\epsilon > 0$ we select a neighborhood U of $e \in G$ such that $|f(g) - f(g')| < \epsilon$ for all $g, g' \in G$ such that $g' \in gU$. We can also assume that $U = U^{-1}$. Then we select a neighborhood V of e such that the closure of V is contained in U. Now choose a nonnegative function ψ such that $\psi = 1$ on V and $\psi = 0$ outside U. From ψ we construct $\chi = c(\psi(g) + \psi(g^{-1}))$ such that $\int_G \chi(g)\, dg = 1$.

Now we define $\phi(g)$ as

$$\phi(g) = \int_G f(g_1)\chi(gg_1^{-1})\, dg_1$$

Setting $h = g_1 g^{-1}$ and using the invariance of the integral gives

$$\phi(g) = \int_G f(hg)\chi(h^{-1})\, dh$$

and from

$$1 = \int_G \chi(g)\, dg = \int_G \chi(h^{-1})\, dh$$

we get

$$f(g) = \int_G f(g)\chi(h^{-1})\, dh.$$

$\phi(g)$ is a function of the type defined in 5.10 and we will show that $|f(g) - \phi(g)| < \epsilon$:

$$\begin{aligned} |f(g) - \phi(g)| &= \left| \int_G f(g)\chi(h^{-1})\, dh - \int_G f(hg)\chi(h^{-1})\, dh \right| \\ &= \left| \int_G (f(g) - f(hg))\, \chi(h^{-1})\, dh \right| \\ &\leq \int_G |(f(g) - f(hg))|\, \chi(h^{-1})\, dh \\ &= \int_U |(f(g) - f(hg))|\, \chi(h^{-1})\, dh \\ &< \int_U \epsilon\chi(h^{-1})\, dh = \epsilon \end{aligned} \tag{5.14}$$

To complete the proof we notice that the set of continuous functions on G is dense in $L^2(G)$, i.e. for every $\epsilon > 0$ and every $f \in L^2(G)$ we can find a continuous function $\tilde{f}$ such that $\left\| f - \tilde{f} \right\| < \epsilon/2$. For $\tilde{f}$ we can find a linear combination ϕ of matrix entries such that $\left\| \tilde{f} - \phi \right\| < \epsilon/2$ which gives the desired approximation of f by a finite sum of matrix entries.

As a simple consequence of this theorem we get the following expansion of functions in $L^2(G)$:

Theorem 5.5 1. Every function $f \in L^2(G)$ can be written in the form:

$$f = \sum_{\alpha\in A} \sum_{l,j} \left\langle f, e_{lj}^{\alpha} \right\rangle e_{lj}^{\alpha} \tag{5.15}$$

$$= \sum_{\alpha\in A} \sum_{l,j} n_\alpha \left\langle f, t_{lj}^{\alpha} \right\rangle t_{lj}^{\alpha} \tag{5.16}$$

2. For every function $f \in L^2(G)$ we have the *Plancerel formula:*

$$\langle f, f \rangle = \sum_{\alpha\in A} \sum_{l,j} \left| \left\langle f, e_{lj}^{\alpha} \right\rangle \right|^2 \tag{5.17}$$

$$= \sum_{\alpha\in A} \sum_{l,j} n_\alpha \left| \left\langle f, t_{lj}^{\alpha} \right\rangle \right|^2 \tag{5.18}$$

We will only mention that in a sense also the converse of the Peter-Weyl theorem is true (for a proof see [37].)

Theorem 5.6 Let $\{T^\alpha, \alpha \in A\}$ be a set of irreducible, unitary representations of a compact group G. Let $\left\{t_{ij}^{\alpha}, \alpha \in A, i, j = 1, ..., \dim T^\alpha\right\}$ be a set of matrix elements of the representations T^α in some orthonormal basis. If the finite linear combinations of these matrix elements form a dense subset of the space $L^2(G)$ (or of the space $C(G)$ of continuous functions on G), then every irreducible, unitary representation of G is unitarily equivalent to one of the representations in $\left\{t_{ij}^{\alpha}, \alpha \in A, i, j = 1, ..., \dim T^\alpha\right\}$.

5.2 Characters

The Peter-Weyl theorem and the characters of a group can be used to find a characterization of the irreducible representations of a compact group. We recall first the definition of a group character:

Let T be a finite-dimensional representation of a compact group G and let $e_1, ...e_n$ be a basis of the representation space H. The matrix elements of T in this basis will be denoted by $t_{ij}(g)$. The character of the representation T is defined as the trace of the representation matrices:

$$\chi(g) = \sum_{i=1}^{n} t_{ii}(g) = \text{tr } (t_{ij}(g)) \tag{5.19}$$

The character of a representation depends only on the representation and not on the selected basis e_i, since the trace of a matrix is independent of the basis of the underlying space. From the properties of the trace it follows also that the character does not change if we replace T by an equivalent representation and that the character is constant on conjugacy classes in G. For the characters we find the following *orthogonality relations for characters:*

Theorem 5.7 The characters χ^α and $\chi^{\alpha'}$ of irreducible unitary representations T^α and $T^{\alpha'}$ of the compact group G satisfy the orthogonality relations:

$$\left\langle \chi^\alpha, \chi^{\alpha'} \right\rangle = \int_G \chi^\alpha(g)\overline{\chi^{\alpha'}(g)}\, dg = \begin{cases} 0 & \text{if } T^\alpha \text{ is not equivalent } T^{\alpha'} \\ 1 & \text{if } T^\alpha \text{ is equivalent } T^{\alpha'} \end{cases}$$

This can be easily seen by inserting the definition of a character in the integral and using the orthogonality of the matrix elements.

For the character of a finite-dimensional unitary representation we get the following characterization:

Theorem 5.8 Let S be a finite-dimensional, unitary representation and assume that S is the direct sum of the irreducible, unitary representations $T_k^\alpha, k = 1, ..., p:$

$$S = n_1 T^{\alpha_1} + ... + n_p T^{\alpha_p}. \tag{5.20}$$

with natural numbers n_k. Denote the character of S by χ and the character of T_k^α by χ_k. Then we have

$$\chi = n_1\chi_1 + ... + n_p\chi_p \tag{5.21}$$

with

$$n_k = \langle \chi, \chi_p \rangle \tag{5.22}$$

Conversely if the character χ of a representation allows a decomposition of the type shown in equation 5.21 then S has a decomposition of the form 5.20.

Theorem 5.9 The characters of two finite-dimensional unitary representations of a compact group G coincide if and only if the representations are equivalent.

Proof: Denote the characters by T^1 and T^2 and the characters by χ^1 and χ^2. If T^1 and T^2 are equivalent then are the characters identical. Conversely if $\chi^1 = \chi^2$ then the decompositions of type 5.21 coincide and so we find that also the representations coincide.

Theorem 5.10 A continuous, finite-dimensional, unitary representation of a compact group G is irreducible if and only if $\int_G |\chi(g)|^2 \, dg = 1$.

Proof: If χ is irreducible then we get $\int_G |\chi(g)|^2 \, dg = 1$ from theorem 5.3. Now assume $\int_G |\chi(g)|^2 \, dg = 1$ then we find also from theorem 5.3

$$1 = \langle \chi, \chi \rangle = n_1^2 \langle \chi_1, \chi_1 \rangle + \dots + n_k^2 \langle \chi_k, \chi_k \rangle = n_1^2 + \dots + n_k^2$$

for natural numbers n_k and characters of irreducible representations.

We will now apply these results to show that the system of irreducible representations of $SO(3)$ that was constructed in the last chapter is indeed complete.

In the beginning of this section we mentioned already that the character of a representation is constant on conjugacy classes. In the case of $SO(3)$ this means that $\chi(g)$ is only a function of the rotation angle of g since we can consider $h^{-1}gh$ instead of g and select h such that h exchanges the rotation axis of g with the z-axis. It can be easily seen that g and $h^{-1}gh$ have the same rotation angle and that $h^{-1}gh$ is a rotation around the z-axis. We conclude that it is sufficient to compute the character value only for rotations around the z-axis. If we consider the representation in the space of surface harmonics $Y_l^{-l}, \dots, Y_l^l$ then we find from the form of the surface harmonics that the character χ_l of this representation has the form:

$$\chi_l(g) = \chi_l(\alpha) = \sum_{m=-l}^{l} e^{im\alpha} = \frac{\sin(l + 1/2)\alpha}{\sin \alpha/2} \tag{5.23}$$

where α is the rotation angle of the rotation g (the last equation is obtained by using the formula for an geometric sum). For the product of two characters χ_l and χ_m we find

$$\chi_l(g)\chi_m(g) = \frac{\cos(l-m)\alpha - \cos(l+m+1)\alpha}{1 - \cos\alpha} \tag{5.24}$$

The functions χ_l are orthonormal on the group $SO(3)$ and we can therefore rewrite the Haar integral for the characters as

$$\int_{SO(3)} \chi_l(g)\chi_m(g) \, dg = \int_0^{\pi} \chi_l(\alpha)\chi_m(\alpha) w(\alpha) \, d\alpha \tag{5.25}$$

for some weight function $w(\alpha)$. From the orthonormality of the characters one gets:

$$w(\alpha) = \frac{1 - \cos\alpha}{\pi} \tag{5.26}$$

and the Haar integral in equation 5.25 becomes thus:

$$\int_{SO(3)} \chi_l(g)\chi_m(g) \, dg = \int_0^{\pi} \chi_l(\alpha)\chi_m(\alpha) \frac{1 - \cos\alpha}{\pi} \, d\alpha \tag{5.27}$$

We use equation 5.27 now to show that the representations in the spaces $Y_l^{-l}, \ldots, Y_l^l$ are indeed a complete set of finite-dimensional, irreducible, unitary representations of $SO(3)$.

If χ is the character of any finite-dimensional irreducible unitary representation of $SO(3)$ then we have $\langle \chi, \chi \rangle = 1$. If χ is different from all χ_l then we have also $\langle \chi, \chi_l \rangle = 0$ for all l. To see that the surface harmonics form indeed a complete set of irreducible representations we have to show that this is impossible:

We use $1 - \cos\alpha = 2(\sin\frac{1}{2}\alpha)^2$ and get:

$$\begin{aligned} \langle \chi, \chi_l \rangle &= \int_0^\pi \chi(\alpha)\chi_l(\alpha)\frac{1-\cos\alpha}{\pi}\, d\alpha \\ &= \frac{2}{\pi}\int_0^\pi \chi(\alpha)\frac{\sin(l+1/2)\alpha}{\sin\alpha/2}(\sin\alpha/2)^2\, d\alpha \\ &= \frac{2}{\pi}\int_0^\pi \chi(\alpha)\sin((l+1/2)\alpha)\sin(\alpha/2)\, d\alpha \end{aligned} \tag{5.28}$$

Making the substitution $t = \alpha/2$ and $F(t) = \sin\frac{\alpha}{2}\chi(\alpha)$ we find

$$\langle \chi, \chi_l \rangle = \frac{4}{\pi}\int_0^{\pi/2} F(t)\sin(2l+1)t\, dt \tag{5.29}$$

Next define $F^-(t)$ as:

$$F^-(t) = \begin{cases} F(t) & \text{if } 0 \le t \le \pi/2 \\ F(t-\pi/2) & \text{if } \pi/2 \le t \le \pi/2 \\ -F^-(t) & \text{if } 0 \le -t \le \pi. \end{cases} \tag{5.30}$$

Then F^- is odd around the origin and even around the points $\pi/2$ and $-\pi/2$. Furthermore F^- is orthogonal to all functions $\sin(2l+1)t$ and we find therefore that F^- is identically zero. F is thus identically zero and thus also χ. This shows that the representations in the space of the surface harmonics exhaust all the finite-dimensional representations of $SO(3)$. Using the fact that all irreducible unitary representations of a compact group are finite-dimensional (see 3.12) we see that we found all irreducible unitary representations of $SO(3)$.

The complex exponentials and the surface harmonics form thus complete sets of basic functions for the functions on the unit circle and the unit sphere respectively:

Theorem 5.11
1. The functions $e^{in\phi}$ form a complete orthonormal system for the square-integrable functions on the unit circle.
2. The surface harmonics Y_l^m form a complete orthonormal system for the square-integrable functions on the unit sphere.

The proof is very simple: Assume f is a function on the circle (sphere) that is orthonormal to all complex exponentials (surface harmonics). Then we consider the spaces $\{f(g(x))|g \in SO(2)\}$ and $\{f(g(x))|g \in SO(3)\}$ respectively. These spaces are invariant under the rotation groups and orthogonal to all exponentials (surface harmonics). These spaces define thus a representation of the group and they contain thus finite-dimensional subspaces that are invariant under the group operation. Since these subspaces are orthogonal to all exponentials (surface harmonics) we see that we have found a new, irreducible, unitary finite-dimensional representation of the rotation group. This is impossible and therefore we have $f = 0$.