

# Bibliography

- [1] H. Arsenault and C. Delisle. Contrast-invariant pattern recognition using circular components. *Applied Optics*, 24(14):2072–2075, 1985.
- [2] H. Arsenault and Y. Sheng. Modified composite filter for pattern recognition in the presence of noise with non-zero mean. *Optics Communications*, 63(1):15–20, 1987.
- [3] J. Babaud, A. P. Witkin, M. Baudin, and R. O. Duda. Uniqueness of the gaussian kernel for scale-space filtering. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, PAMI-8(1):26–33, 1986.
- [4] L. Breiman. *Probability*. Addison-Wesley, Reading, 1968.
- [5] C. Chevalley. *Theory of Lie-Groups*. Princeton University Press, 1946.
- [6] P. E. Danielsson. Rotation invariant linear operators with directional response. In *Proc. 5. ICPR*, pages 1171–1176, 1980.
- [7] P. E. Danielsson. Natural basis functions for image analysis. In D. Meyer-Ebrecht, editor, *Proceedings of the 6. Aachener Symposium für Signaltheorie*, Informatik-Fachberichte, Vol. 153, pages 239–254, Springer-Verlag, Berlin, Heidelberg, New York, 1987.
- [8] P. E. Danielsson and Henrik Sauleda. Rotation invariant 2d filters matched to 1d features. In *Proceedings of the IEEE Computer Society Conference on Computer Vision and Pattern Recognition, San Francisco*, 1985.
- [9] S. R. Deans. *The Radon Transform and Some of its Applications*. Wiley Interscience, 1983.
- [10] N. Dunford and J. T. Schwartz. *Linear Operators (Vols. I, II)*. Interscience Publishers, New York, 1958, 1963.
- [11] A. Erdelyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi. *Higher Transcendental Functions*. McGraw-Hill, New York, Toronto, London, 1953.
- [12] Rumelhart et al. Learning representation by back-propagation errors. *Nature*, 323(6088):5533, 1986.
- [13] Rumelhart et al. *Parallel Distributed Processing (Vols. 1,2)*. MIT Press, Cambridge, 1986.

- [14] I. M. Gelfand, M. I. Graev, and N. Y. Vilenkin. *Generalized Functions, Vol. 5 (Integral Geometry and Representation Theory)*. Academic Press, 1966.
- [15] I. M. Gelfand, R. A. Minlos, and Z. Y. Shapiro. *Representations of the rotation and Lorentz groups and their applications*. Pergamon Press, 1963.
- [16] J. Goodman. *Introduction to Fourier Optics*. Physical and Quantum Electronics Series. McGraw-Hill, 1968.
- [17] P.R. Halmos, editor. *Measure Theory*. D. Van Nostrand, Reading, Mass., 1950.
- [18] H.D.Ebbinghaus, H. Hermes, F. Hirzebruch, M. Koecher, K. Mainzer, J. Neukirch, A. Prestel, and R. Remmert. *Zahlen*. Springer-Verlag, Berlin, Heidelberg, New York, 1988.
- [19] F. Hirzebruch, W. Scharlau. *Einführung in die Funktionalanalysis*. Bibliographisches ches Institut, Mannheim, Wien, Zürich, 1971.
- [20] R. A. Hummel. Feature detection using basis functions. *Computer Graphics and Image Processing*, 9:40–55, 1979.
- [21] S. H. Izen. A series inversion for the x-ray transform in n dimensions. *Inverse Problems*, 4:725–748, 1988.
- [22] K. I. Kanatani. Camera rotation invariance of image characteristics. *Computer Vision, Graphics and Image Processing*, 39:328–354, 1987.
- [23] K. I. Kanatani. Transformation of optical flow by camera rotation. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 10(2):131–143, 1988.
- [24] T. Kohonen. *Self organization and associative memory*. Springer Series in Information Sciences, Vol. 8. Springer-Verlag, Berlin, Heidelberg, New York, 1984.
- [25] S. Lang. *Algebra*. Addison-Wesley, 1965.
- [26] G. Laub and R. Bachus. Visualization of vessels with mri. In *Proc. Computer Aided Radiology*. Springer-Verlag, Berlin, Heidelberg, New York, 1987.
- [27] R. Lenz. *Reconstruction, Processing and Display of 3D-Images*. PhD thesis, Linköping University, Sweden, 1986.
- [28] R. Lenz. Optimal filters for the detection of linear patterns in 2-d and higher dimensional images. *Pattern Recognition*, 20(2):163–172, 1987.
- [29] R. Lenz. Rotation-invariant operators and scale-space filtering. *Pattern Recognition Letters*, 6:151–154, 1987.
- [30] R. Lenz. A group theoretical approach to filter design. In *Proc. International Conference on Acoustics, Speech and Signal Processing*, 1989.
- [31] R. Lenz. A group theoretical model of feature extraction. *Journal of the Optical Society of America A*, 6(6):827–834, June 1989.

- [32] M. P. Levesque and H. H. Arsenault. Rotation-invariant pattern recognition using the phase of the circular harmonic filter correlations. *Optics Communications*, 58(3):161–166, 1986.
- [33] R. Lippman. An introduction to computing with neural nets. *IEEE ASSP Magazine*, 4:4–22, 1987.
- [34] D. Marr. *Vision*. W. H. Freeman, 1982.
- [35] M. Minsky and S. Papert. *Perceptrons*. MIT Press, Cambridge, 1969.
- [36] M. A. Naimark. *Linear Representations of the Lorentz Group*. Pergamon Press, Oxford, London, 1964.
- [37] M. A. Naimark and A. I. Stern. *Theory of Group Representations*. Springer-Verlag, New York, Berlin, Heidelberg, 1982.
- [38] A. Papoulis. *System and Transforms with Applications in Optics*. McGraw-Hill, New York, 1968.
- [39] W. Pitts and W. S. McCulloch. How we know universals: the perception of auditory and visual forms. *Bulletin of Mathematical Biophysics*, 9:127–147, 1947.
- [40] L. Pontrjagin. *Topological Groups*. Princeton Mathematical Series No. 2, 1946.
- [41] F. Riesz and B. Sz. Nagy. *Vorlesungen über Funktionalanalysis*. Deutscher Verlag der Wissenschaften, Berlin, 1956.
- [42] Y. Sheng and H. H. Arsenault. Experiments on pattern recognition using invariant fourier- mellin descriptors. *Journal of the Optical Society of America A*, 3(6):771–776, 1986.
- [43] Y. Sheng and J. Duvernoy. Circular-fourier-radial-mellin transform descriptors for pattern recognition. *Journal of the Optical Society of America A*, 3(6):885–888, 1986.
- [44] D. Slepian. Prolate spheroidal wave functions, fourier analysis and uncertainty - iv: Extensions to many dimensions; generalized prolate spheroidal functions. *Bell System Technical Journal*, 43:3009–3058, 1964.
- [45] M. Sugiura. *Unitary Representations and Harmonic Analysis*. Kodansha Ltd., Tokyo, 1975.
- [46] A. Terras. *Harmonic Analysis on Symmetric Spaces and Applications I*. Springer-Verlag, Berlin, Heidelberg, New York, 1985.
- [47] V. S. Varadarajan. *Lie Groups, Lie Algebras and their Representations*. Prentice-Hall, Englewood Cliffs, N.J., 1974.
- [48] N. Ya. Vilenkin. *Special Functions and the Theory of Group Representations*. AMS Translations. American Mathematical Society, Providence, R.I., 1968.
- [49] V. S. Vladimirov. *Equations of Mathematical Physics*. Mir Publishers, Moscow, 1984.

- [50] G. N. Watson. *A Treatise on the Theory of Bessel Functions*. Cambridge University Press, 1966.
- [51] R. Wu and H. Stark. Rotation-invariant pattern recognition using a vector reference. *Applied Optics*, 23(6):838–840, 1984.
- [52] R. Wu and H. Stark. Rotation-invariant pattern recognition using optimum feature extraction. *Applied Optics*, 24(2):179–184, 1985.
- [53] K. Yosida. *Functional Analysis*. Springer-Verlag, Berlin, Heidelberg, New York, 1978.
- [54] A. L. Yuille and T. Poggio. Scaling theorems for zero crossings. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, PAMI-8(1):15–25, 1986.
- [55] D. P. Zelobenko. *Compact Lie Groups and their Representations*. American Mathematical Society, Providence, R. I., 1973.
- [56] S. W. Zucker and R. A. Hummel. A three-dimensional edge operator. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, PAMI-3(3):324–331, 1981.

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