Applying Geometric Thick Paths to Compute the Number of Additional Train Paths in a Railway Timetable

Anders Peterson                      Valentin Polishchuk                      Christiane Schmidt
Introduction

Routing a Maximum Number of Thick Paths through a Polygonal Domain

Thick Paths with Limited Slope

Construction of Polygonal Domain from the Timetable

Example

Conclusion and Outlook
• Marshalling yards: completed trains occupy highly demanded space until departure
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- We:
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  - We present optimal solution
Existing trains
Additional trains: need to keep temporal distance
Existing trains

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⇒ Thick paths instead of lines
Existing trains

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→ Thick paths instead of lines
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We start with 2
Routing a Maximum Number of Thick Paths through a Polygonal Domain
Simple polygon P
Simple polygon P

Holes
Polygonal domain

Simple polygon $P$

Holes
Simple polygon $P$

Holes

Source $\Gamma_s$

Polygonal domain
Simple polygon $P$

Holes

Source $\Gamma_s$

Sink $\Gamma_t$
Simple polygon $P$

Polygonsal domain

Source $\Gamma_s$

Bottom

Sink $\Gamma_t$
Route thick paths from the source to the sink, avoiding all holes (=obstacles)
Thin path $\pi$: simple curve
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Let $C_r$ denote the open disk of radius $r$ centered at the origin
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For $S \subset \mathbb{R}^2$: $(S)^r = S \oplus C_r = \{x+y | x \in S, y \in C_r\}$ - Minkowski sum
Thick path $\Pi$: Minkowski sum of a thin path and a unit disk $\Pi = (\pi)^1$
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- Need some more concepts ($\Omega$ perforated at the source and sinks and Riemann flaps glued to $\Omega$, ...)
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- Some additional tweaks when we hit a hole after $\tau<2$
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Thick Paths with Limited Slope
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- We showed how to adapt the waterfall construction to compute the maximum number of thick non-crossing paths with a given slope range (\( \triangleq C \)-respecting)
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**Theorem:** A representation of the maximum number of $C$-respecting thick-non-crossing paths can be found in $O(nh+n\log n)$ time.
• We consider the time-space diagram—the geometric representation
• Inserting a new train: Route path from start to end station
• Paths not arbitrarily close ⇒ temporal distance (different to trains running in same or opposite direction)
• We think of train paths as “blown-up” line segments = thick paths
• Blown up by temporal distance (can be minimum, or more)
• How to route those thick paths? Concepts from Computational Geometry
• Need to make some adaptations, for example, if stations are lines, no path could cross these
  ➡ 1. Show how to construct the appropriate polygonal domain
  ➡ 2. Show how to route the maximum number of thick non-crossing paths in that domain:
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Still left to do
Construction of Polygonal Domain from the Timetable
If we would define the time windows as source and sink

cone: \[ \big/ \]
thick path: \[ \big/ \]
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cone: \( w \)
thick path: \( w \)
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⇒ Possible thick paths would correspond to train paths in a smaller time interval
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⇒ Extend the time windows by $d/2$ to both sides to create $\Gamma_s$ and $\Gamma_t$
($\Gamma_s=[p_1,p_2], \Gamma_t=[p_3,p_4]$)
If we would define the time windows as source and sink
⇒ Possible thick paths would correspond to train paths in a smaller time interval
⇒ Extend the time windows by \( d/2 \) to both sides to create \( \Gamma_s \) and \( \Gamma_t \)
\( (\Gamma_s = [p_1, p_2], \Gamma_t = [p_3, p_4]) \)
$s_i$  \hspace{1cm} $s_{i+1}$

$\angle$ allowed cone
Vertical lines at stations are obstacles
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⇒ “Cut” each station open and blow up by vertical distance:
  - If the station $s$ has exactly $k$ sidetracks, we insert a vertical distance of $k \cdot d$
  - If no such limit exists, we can insert a vertical distance of $\min\{|\Gamma_s| + d, |\Gamma_t| + d\}$
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⇒ We need to shift the consecutive stations to the right, such that this path can be reached with limited slope
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In the example we used \(d_s=d, \, d_0=d/2\)
cone: \( / \)

thick path:
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$\Rightarrow \ell_2$
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  \[
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- No train can run earlier than departing earliest with highest speed
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- No train can run later than arriving latest with highest speed
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- Some further boundary parts
- Intersect holes with boundary
Example
cone: \begin{center}
\begin{tikzpicture}
\fill[fill=teal, thick] (-1,-1) -- (-1,1) -- (1,1) -- (1,-1) -- cycle;
\end{tikzpicture}
\end{center}

thick path:
cone: \[\text{thick path:}\]
cone:

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cone: 

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\_\_\_
Conclusion and Outlook
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  - NP-hard in general
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• Application to real-world example
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Outlook

• Application to real-world example
• What other geometric concepts can be used?